CONIC SECTIONS

11.1 Overview

11.1.1 *Sections of a cone* Let *l* be a fixed vertical line and *m* be another line intersecting it at a fixed point V and inclined to it at an angle α (Fig. 11.1).
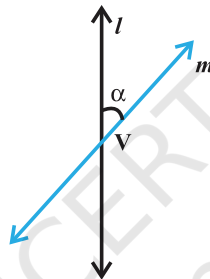

Fig. 11.1

Suppose we rotate the line *m* around the line *l* in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone herein after referred as cone and extending indefinitely in both directions (Fig. 11.2).
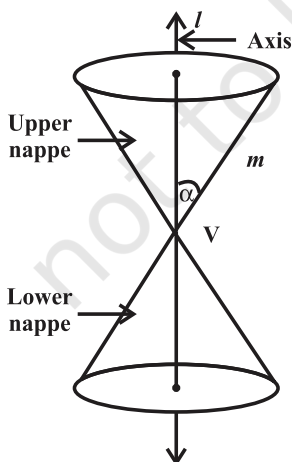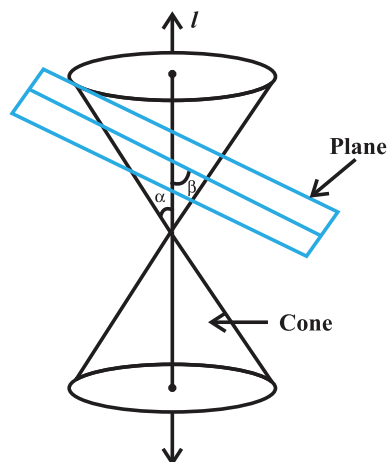

Fig. 11.2

Fig. 11.3

The point V is called the *vertex*; the line *l* is the *axis* of the cone. The rotating line *m* is called a *generator* of the cone. The *vertex* separates the cone into two parts called *nappes*.

If we take the intersection of a plane with a cone, the section so obtained is called a *conic section*. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane.

We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and the angle made by it with the vertical axis of the cone. Let β be the angle made by the intersecting plane with the vertical axis of the cone (Fig.11.3).

The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex.

When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations:

(a)   When β = 90°, the section is a *circle*.

(b)   When α < β < 90°, the section is an *ellipse*.

(c)   When β = α; the section is a *parabola*.

      (In each of the above three situations, the plane cuts entirely across one nappe of the cone).

(d)   When 0 ≤ β < α; the plane cuts through both the nappes and the curves of intersection is a *hyperbola*.

Indeed these curves are important tools for present day exploration of outer space and also for research into the behaviour of atomic particles.

We take conic sections as plane curves. For this purpose, it is convenient to use equivalent definition that refer only to the plane in which the curve lies, and refer to special points and lines in this plane called foci and directrices. According to this approach, parabola, ellipse and hyperbola are defined in terms of a fixed point (called focus) and fixed line (called directrix) in the plane.

If S is the focus and *l* is the directrix, then the set of all points in the plane whose distance from S bears a constant ratio *e* called eccentricity to their distance from *l* is a conic section.

As special case of ellipse, we obtain circle for which *e* = 0 and hence we study it differently.

11.1.2 *Circle* A circle is the set of all points in a plane which are at a fixed distance from a fixed point in the plane. The fixed point is called the centre of the circle and the distance from centre to any point on the circle is called the radius of the circle.

The equation of a circle with radius r having centre (h, k) is given by $(x – h)^2 + (y – k)^2 = r^2$

The general equation of the circle is given by $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f and c are constants.

(a)   The centre of this circle is $(– g, – f)$

(b)   The radius of the circle is $\sqrt{g^2 + f^2 - c}$

The general equation of the circle passing through the origin is given by $x^2 + y^2 + 2gx + 2fy = 0$.

Fig. 11.4

General equation of second degree i.e., $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a circle if (i) the coefficient of x^2 equals the coefficient of y^2, i.e., $a = b \neq 0$ and (ii) the coefficient of xy is zero, i.e., $h = 0$.

The parametric equations of the circle $x^2 + y^2 = r^2$ are given by $x = r \cos\theta, y = r \sin\theta$ where θ is the parameter and the parametric equations of the circle $(x – h)^2 + (y – k)^2 = r^2$ are given by

$$x – h = r \cos\theta, \ y – k = r \sin\theta$$

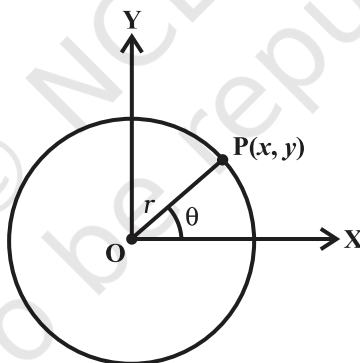or $$x = h + r \cos\theta, \ y = k + r \sin\theta.$$

Fig. 11.5

Note: The general equation of the circle involves three constants which implies that at least three conditions are required to determine a circle uniquely.

11.1.3 *Parabola*

A parabola is the set of points P whose distances from a fixed point F in the plane are equal to their distances from a fixed line l in the plane. The fixed point F is called focus and the fixed line l the **directrix** of the parabola.
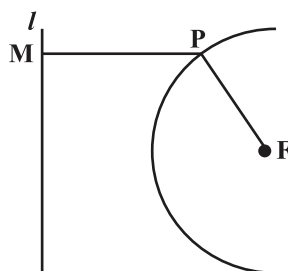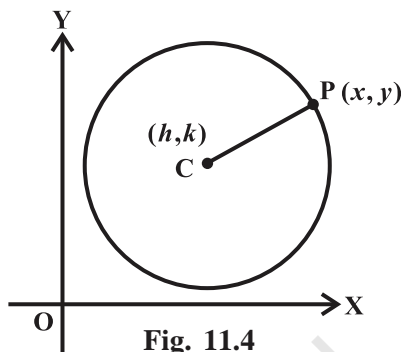

Fig. 11.6

Standard equations of parabola

The four possible forms of parabola are shown below in Fig. 11.7 (a) to (d)

The latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola (Fig. 11.7).

(a) $y^2 = 4ax$ **(b)** $y^2 = -4ax$

(c) $x^2 = 4ay$ **(d)** $x^2 = -4ay$
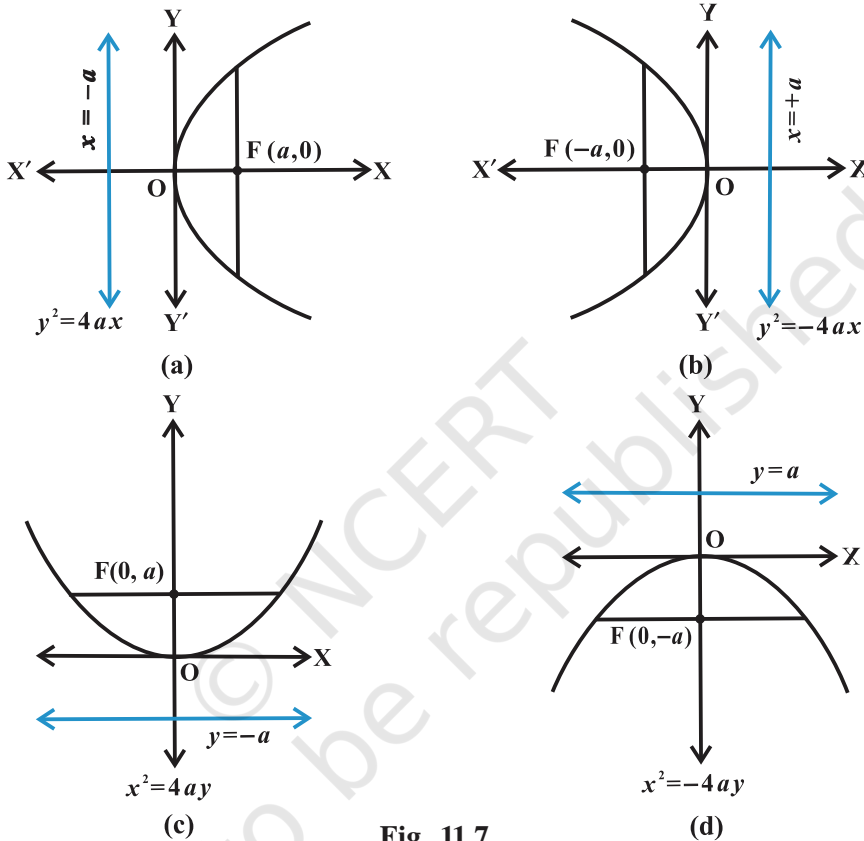
Fig. 11.7

Main facts about the parabola

Forms of Parabolas	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Equations of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$

Focal distance of a point

Let the equation of the parabola be $y^2 = 4ax$ and P(x, y) be a point on it. Then the distance of P from the focus $(a, 0)$ is called the focal distance of the point, i.e.,

$$\begin{aligned} \text{FP} &= \sqrt{(x-a)^2 + y^2} \\ &= \sqrt{(x-a)^2 + 4ax} \\ &= \sqrt{(x+a)^2} \\ &= |x + a| \end{aligned}$$

11.1.4 *Ellipse*   An ellipse is the set of points in a plane, the sum of whose distances from two fixed points is constant. Alternatively, an ellipse is the set of all points in the plane whose distances from a fixed point in the plane bears a constant ratio, less than, to their distance from a fixed line in the plane. The fixed point is called focus, the fixed line a directrix and the constant ratio (e) the centricity of the ellipse.

We have two standard forms of the ellipse, i.e.,

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and   (ii) $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$,

In both cases $a > b$ and $b^2 = a^2(1 - e^2)$, $e < 1$.

In (i) major axis is along x-axis and minor along y-axis and in (ii) major axis is along y-axis and minor along x-axis as shown in Fig. 11.8 (a) and (b) respectively.
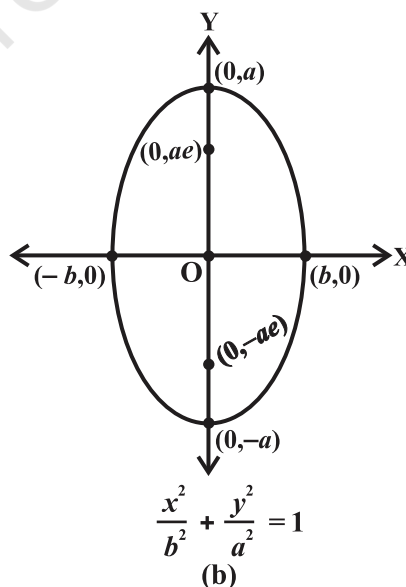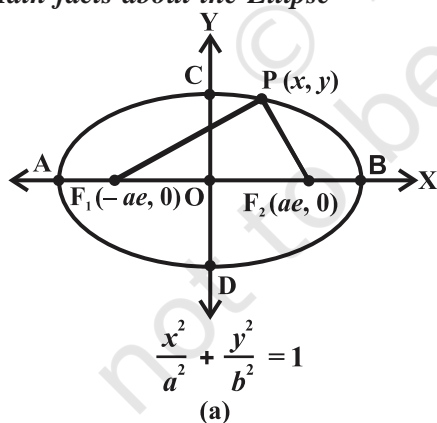
Main facts about the Ellipse

Fig. 11.8

Forms of the ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$
	$a > b$	$a > b$
Equation of major axis	$y = 0$	$x = 0$
Length of major axis	$2a$	$2a$
Equation of Minor axis	$x = 0$	$y = 0$
Length of Minor axis	$2b$	$2b$
Directrices	$x = \pm\ \dfrac{a}{e}$	$y = \pm\ \dfrac{a}{e}$
Equation of latus rectum	$x = \pm\ ae$	$y = \pm\ ae$
Length of latus rectum	$\dfrac{2b^2}{a}$	$\dfrac{2b^2}{a}$
Centre	$(0, 0)$	$(0, 0)$

Focal Distance

The focal distance of a point (x, y) on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$a - e \mid x \mid \text{ from the nearer focus}$$
$$a + e \mid x \mid \text{ from the farther focus}$$

Sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis.

11.1.5 *Hyperbola* A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points is constant. Alternatively, a hyperbola is the set of all points in a plane whose distances from a fixed point in the plane bears a constant ratio, greater than 1, to their distances from a fixed line in the plane. The fixed point is called a focus, the fixed line a directrix and the constant ratio denoted by e, the ecentricity of the hyperbola.

We have two standard forms of the hyperbola, i.e.,

(i) $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad$ and $\qquad\qquad$ (ii) $\quad \dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

Here $b^2 = a^2 (e^2 - 1)$, $e > 1$.

In (i) transverse axis is along x-axis and conjugate axis along y-axis where as in (ii) transverse axis is along y-axis and conjugate axis along x-axis.
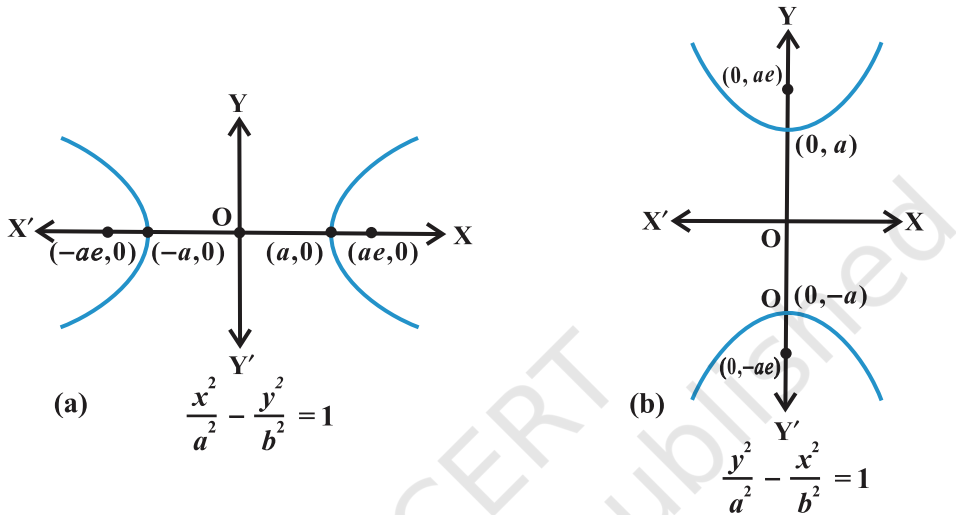
Fig. 11.9

Main facts about the Hyperbola

Forms of the hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$
Length of transverse axis	$2a$	$2a$
Foci	$(\pm\, ae, 0)$	$(0, \pm\, ae)$
Equation of latus rectum	$x = \pm\, ae$	$y = \pm\, ae$
Length of latus rectum	$\dfrac{2b^2}{a}$	$\dfrac{2b^2}{a}$
Centre	$(0, 0)$	$(0, 0)$

Focal distance

The focal distance of any point (x, y) on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$e \mid x \mid - a$ from the nearer focus

$e \mid x \mid + a$ from the farther focus

Differences of the focal distances of any point on a hyperbola is constant and equal to the length of the transverse axis.

Parametric equation of conics

Conics	Parametric equations
(i) Parabola : $y^2 = 4ax$	$x = at^2$, $y = 2at$; $-\infty < t < \infty$
(ii) Ellipse : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x = a\cos\theta$, $y = b\sin\theta$; $0 \le \theta \le 2\pi$
(iii) Hyperbola : $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$x = a\sec\theta$, $y = b\tan\theta$, where

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}; \ \frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

11.2 Solved Examples

Short Answer Type

Example 1 Find the centre and radius of the circle $x^2 + y^2 - 2x + 4y = 8$

Solution we write the given equation in the form $(x^2 - 2x) + (y^2 + 4y) = 8$

Now, completing the squares, we get

$(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 1 + 4$

$(x - 1)^2 + (y + 2)^2 = 13$

Comparing it with the standard form of the equation of the circle, we see that the centre of the circle is $(1, -2)$ and radius is $\sqrt{13}$.

Example 2 If the equation of the parabola is $x^2 = -8y$, find coordinates of the focus, the equation of the directrix and length of latus rectum.

Solution The given equation is of the form $x^2 = -4ay$ where a is positive.

Therefore, the focus is on y-axis in the negative direction and parabola opens downwards.

Comparing the given equation with standard form, we get $a = 2$.

Therefore, the coordinates of the focus are $(0, -2)$ and the the equation of directrix is $y = 2$ and the length of the latus rectum is $4a$, i.e., 8.

Example 3 Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the major and minor axes, eccentricity, foci and vertices.

Solution We put the equation in standard form by dividing by 225 and get

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

This shows that $a = 5$ and $b = 3$. Hence $9 = 25(1 - e^2)$, so $e = \dfrac{4}{5}$. Since the denominator of x^2 is larger, the major axis is along x-axis, minor axis along y-axis, foci are $(4, 0)$ and $(-4, 0)$ and vertices are $(5, 0)$ and $(-5, 0)$.

Example 4 Find the equation of the ellipse with foci at $(\pm 5, 0)$ and $x = \dfrac{36}{5}$ as one of the directrices.

Solution We have $ae = 5$, $\dfrac{a}{e} = \dfrac{36}{5}$ which give $a^2 = 36$ or $a = 6$. Therefore, $e = \dfrac{5}{6}$.

Now $b = a\sqrt{1 - e^2} = 6\sqrt{1 - \dfrac{25}{36}} = \sqrt{11}$. Thus, the equation of the ellipse is $\dfrac{x^2}{36} + \dfrac{y^2}{11} = 1$.

Example 5 For the hyperbola $9x^2 - 16y^2 = 144$, find the vertices, foci and eccentricity.

Solution The equation of the hyperbola can be written as $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, so $a = 4$, $b = 3$ and $9 = 16 (e^2 - 1)$, so that $e^2 = \dfrac{9}{16} + 1 = \dfrac{25}{16}$, which gives $e = \dfrac{5}{4}$. Vertices are $(\pm a, 0) = (\pm 4, 0)$ and foci are $(\pm ae, 0) = (\pm 5, 0)$.

Example 6 Find the equation of the hyperbola with vertices at $(0, \pm 6)$ and $e = \dfrac{5}{3}$. Find its foci.

Solution Since the vertices are on the y-axes (with origin at the mid-point), the equation is of the form $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.

As vertices are $(0, \pm 6)$, $a = 6$, $b^2 = a^2 (e^2 - 1) = 36 \left(\dfrac{25}{9} - 1 \right) = 64$, so the required

equation of the hyperbola is $\dfrac{y^2}{36} - \dfrac{x^2}{64} = 1$ and the foci are $(0, \pm ae) = (0, \pm 10)$.

Long Answer Type

Example 7 Find the equation of the circle which passes through the points (20, 3), (19, 8) and (2, –9). Find its centre and radius.

Solution By substitution of coordinates in the general equation of the circle given by $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$$40g + 6f + c = -409$$
$$38g + 16f + c = -425$$
$$4g - 18f + c = -85$$

From these three equations, we get

$g = -7, f = -3$ and $c = -111$

Hence, the equation of the circle is

$$x^2 + y^2 - 14x - 6y - 111 = 0$$

or $\qquad\qquad (x - 7)^2 + (y - 3)^2 = 13^2$

Therefore, the centre of the circle is (7, 3) and radius is 13.

Example 8 An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ whose one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Solution As shown in the figure APQ denotes the equilateral triangle with its equal sides of length l (say).

Here $\qquad\qquad$ AP $=$ l so AR $= l \cos 30°$

$\qquad\qquad\qquad\qquad = l \dfrac{\sqrt{3}}{2}$

Also, $\qquad\qquad$ PR $= l \sin 30° = \dfrac{l}{2}$.

Thus $\left(\dfrac{l\sqrt{3}}{2}, \dfrac{l}{2} \right)$ are the coordinates of the point P lying on the
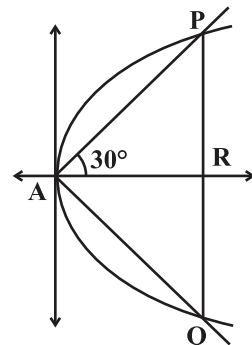
parabola $y^2 = 4ax$.

Fig. 11.10

Therefore, $$\frac{l^2}{4} = 4a\left(\frac{l\sqrt{3}}{2}\right) \Rightarrow l = 8\,a\sqrt{3}.$$

Thus, $8\,a\sqrt{3}$ is the required length of the side of the equilateral triangle inscribed in the parabola $y^2 = 4ax$.

Example 9 Find the equation of the ellipse which passes through the point $(-3, 1)$ and has eccentricity $\dfrac{\sqrt{2}}{5}$, with x-axis as its major axis and centre at the origin.

Solution Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the equation of the ellipse passing through the point $(-3, 1)$.

Therefore, we have $$\frac{9}{a^2} + \frac{1}{b^2} = 1.$$

or $$9b^2 + a^2 = a^2\,b^2$$

or $$9a^2\,(a^2 - e^2) + a^2 = a^2\,a^2\,(1 - e^2) \quad \text{(Using } b^2 = a^2\,(1 - e^2)\text{)}$$

or $$a^2 = \frac{32}{3}$$

Again $$b^2 = a^2\,(1 - e^2) = \frac{32}{3}\left(1 - \frac{2}{5}\right) = \frac{32}{5}$$

Hence, the required equation of the ellipse is

$$\frac{x^2}{\dfrac{32}{3}} + \frac{y^2}{\dfrac{32}{5}} = 1$$

or $$3x^2 + 5y^2 = 32.$$

Example 10 Find the equation of the hyperbola whose vertices are $(\pm\,6, 0)$ and one of the directrices is $x = 4$.

Solution As the vertices are on the x-axis and their middle point is the origin, the equation is of the type $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Here $b^2 = a^2\,(e^2 - 1)$, vertices are $(\pm\,a, 0)$ and directrices are given by $x = \pm\,\dfrac{a}{e}$.

Thus $a = 6$, $\dfrac{a}{e} = 4$ and so $e = \dfrac{3}{2}$ which gives $b^2 = 36\left(\dfrac{9}{4} - 1\right) = 45$

Consequently, the required equation of the hyperbola is $\dfrac{x^2}{36} - \dfrac{y^2}{45} = 1$

Objective Type Questions

Each of the examples from 11 to 16, has four possible options, out of which one is correct. Choose the correct answer from the given four options (M.C.Q.)

Example 11 The equation of the circle in the first quadrant touching each coordinate axis at a distance of one unit from the origin is:

(A) $x^2 + y^2 - 2x - 2y + 1 = 0$　　　　(B) $x^2 + y^2 - 2x - 2y - 1 = 0$

(C) $x^2 + y^2 - 2x - 2y = 0$　　　　(C) $x^2 + y^2 - 2x + 2y - 1 = 0$

Solution The correct choice is (A), since the equation can be written as $(x - 1)^2 + (y - 1)^2 = 1$ which represents a circle touching both the axes with its centre $(1, 1)$ and radius one unit.

Example 12 The equation of the circle having centre $(1, -2)$ and passing through the point of intersection of the lines $3x + y = 14$ and $2x + 5y = 18$ is

(A) $x^2 + y^2 - 2x + 4y - 20 = 0$　　　　(B) $x^2 + y^2 - 2x - 4y - 20 = 0$

(C) $x^2 + y^2 + 2x - 4y - 20 = 0$　　　　(D) $x^2 + y^2 + 2x + 4y - 20 = 0$

Solution The correct option is (A). The point of intersection of $3x + y - 14 = 0$ and $2x + 5y - 18 = 0$ are $x = 4, y = 2$, i.e., the point $(4, 2)$

Therefore, the radius is $= \sqrt{9 + 16} = 5$ and hence the equation of the circle is given by

$$(x - 1)^2 + (y + 2)^2 = 25$$
or 　　　　$x^2 + y^2 - 2x + 4y - 20 = 0.$

Example 13 The area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum is

(A)　12 sq. units

(B)　16 sq. units

(C)　18 sq. units

(D)　24 sq. units

Solution The correct option is (C). From the figure, OPQ represent the triangle whose area is to be determined. The area of the triangle

$= \dfrac{1}{2}\,\text{PQ} \times \text{OF} = \dfrac{1}{2}\,(12 \times 3) = 18$
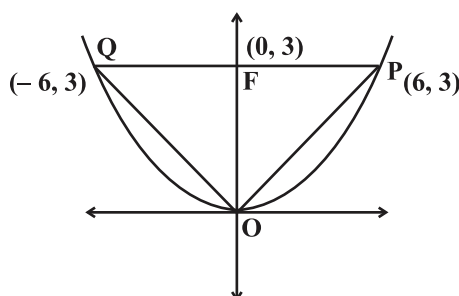

Fig. 11.11

Example 14 The equations of the lines joining the vertex of the parabola $y^2 = 6x$ to the points on it which have abscissa 24 are

    (A) $y \pm 2x = 0$

    (B) $2y \pm x = 0$

    (C) $x \pm 2y = 0$

    (D) $2x \pm y = 0$

Solution   (B) is the correct choice. Let P and Q be points on the parabola $y^2 = 6x$ and OP, OQ be the lines joining the vertex O to the points P and Q whose abscissa are 24.

Fig. 11.12

Thus $y^2 = 6 \times 24 = 144$

or $y = \pm 12$.

Therefore the coordinates of the points P and Q are $(24, 12)$ and $(24, -12)$ respectively. Hence the lines are

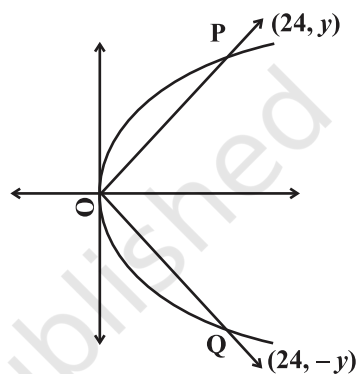$$y = \pm \frac{12}{24}x \Rightarrow 2y = \pm x.$$

Example 15 The equation of the ellipse whose centre is at the origin and the x-axis, the major axis, which passes through the points $(-3, 1)$ and $(2, -2)$ is

    (A) $5x^2 + 3y^2 \ 32$ (B) $3x^2 + 5y^2 = 32$

    (C) $5x^2 - 3y^2 = 32$ (D) $3x^2 + 5y^2 + 32 = 0$

Solution (B) is the correct choice. Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the equation of the ellipse.

Then according to the given conditions, we have

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \qquad \text{and} \qquad \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{4}$$

which gives $a^2 = \dfrac{32}{3}$ and $b^2 = \dfrac{32}{5}$.

Hence, required equation of ellipse is $3x^2 + 5y^2 = 32$.

Example 16 The length of the transverse axis along x-axis with centre at origin of a hyperbola is 7 and it passes through the point $(5, -2)$. The equation of the hyperbola is

(A) $\dfrac{4}{49}x^2 - \dfrac{196}{51}y^2 = 1$

(B) $\dfrac{49}{4}x^2 - \dfrac{51}{196}y^2 = 1$

(C) $\dfrac{4}{49}x^2 - \dfrac{51}{196}y^2 = 1$

(D)   none of these

Solution   (C) is the correct choice. Let $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ represent the hyperbola. Then

according to the given condition, the length of transverse axis, i.e., $2a = 7 \Rightarrow a = \dfrac{7}{2}$.

Also, the point (5, –2) lies on the hyperbola, so, we have

$$\dfrac{4}{49}(25) - \dfrac{4}{b^2} = 1 \quad \text{which gives}$$

$b^2 = \dfrac{196}{51}$. Hence, the equation of the hyperbola is

$$\dfrac{4}{49}x^2 - \dfrac{51}{196}y^2 = 1$$

State whether the statements in Examples 17 and 18 are correct or not. Justify.

Example 17   Circle on which the coordinates of any point are $(2 + 4\cos\theta, -1 + 4\sin\theta)$ where θ is parameter is given by $(x - 2)^2 + (y + 1)^2 = 16$.

Solution True. From given conditions, we have

$$x = 2 + 4\cos\theta \Rightarrow (x - 2) = 4\cos\theta$$

and $y = -1 + 4\sin\theta \Rightarrow y + 1 = 4\sin\theta$.
Squaring

and adding, we get $(x - 2)^2 + (y + 1)^2 = 16$.

Example 18 A bar of given length moves with its extremities on two fixed straight lines at right angles. Any point of the bar describes an ellipse.

Solution True. Let P (x, y) be any point on the bar such that PA = a and PB = b, clearly from the Fig. 11.13.
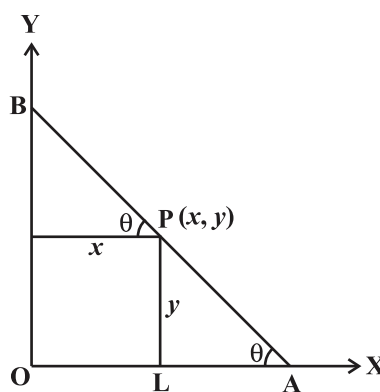

Fig. 11.13

$$x = \text{OL} = b \cos\theta \text{ and}$$
$$y = \text{PL} = a \sin\theta$$

These give $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, which is an ellipse.

Fill in the blanks in Examples 19 to 23.

Example 19 The equation of the circle which passes through the point (4, 5) and has its centre at (2, 2) is _____ .

Solution As the circle is passing through the point (4, 5) and its centre is (2, 2) so its radius is $\sqrt{(4-2)^2 + (5-2)^2} = \sqrt{13}$. Therefore the required answer is $(x-2)^2 + (y-2)^2 = 13$.

Example 20 A circle has radius 3 units and its centre lies on the line $y = x - 1$. If it passes through the point (7, 3), its equation is _____ .

Solution Let (h, k) be the centre of the circle. Then $k = h - 1$. Therefore, the equation of the circle is given by $(x - h)^2 + [y - (h - 1)]^2 = 9$ ... (1)

Given that the circle passes through the point (7, 3) and hence we get

$$(7 - h)^2 + (3 - (h - 1))^2 = 9$$
or
$$(7 - h)^2 + (4 - h)^2 = 9$$
or
$$h^2 - 11h + 28 = 0$$
which gives $(h - 7)(h - 4) = 0$ $\Rightarrow h = 4$ or $h = 7$

Therefore, the required equations of the circles are $x^2 + y^2 - 8x - 6y + 16 = 0$
or
$$x^2 + y^2 - 14x - 12y + 76 = 0$$

Example 21 If the latus rectum of an ellipse with axis along x-axis and centre at origin is 10, distance between foci = length of minor axis, then the equation of the ellipse is _____ .

Solution Given that $\dfrac{2b^2}{a} = 10$ and $2ae = 2b \Rightarrow b = ae$

Again, we know that

$$b^2 = a^2(1 - e^2)$$
or
$$2a^2e^2 = a^2 \Rightarrow e = \frac{1}{\sqrt{2}} \quad (\text{using } b = ae)$$
Thus
$$a = b\sqrt{2}$$

Again
$$\frac{2b^2}{a} = 10$$

or $b = 5\sqrt{2}$. Thus we get $a = 10$

Therefore, the required equation of the ellipse is

$$\frac{x^2}{100} + \frac{y^2}{50} = 1$$

Example 22 The equation of the parabola whose focus is the point (2, 3) and directrix is the line $x - 4y + 3 = 0$ is _____ .

Solution Using the definition of parabola, we have

$$\sqrt{(x-2)^2 + (y-3)^2} = \left| \frac{x - 4y + 3}{\sqrt{17}} \right|$$

Squaring, we get

$$17 (x^2 + y^2 - 4x - 6y + 13) = x^2 + 16y^2 + 9 - 8xy - 24y + 6x$$

or $16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0$

Example 23 The eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which passes through

the points (3, 0) and $(3\sqrt{2}, 2)$ is _____ .

Solution Given that the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is passing through the points (3, 0) and

$(3\sqrt{2}, 2)$, so we get $a^2 = 9$ and $b^2 = 4$.

Again, we know that $b^2 = a^2 (e^2 - 1)$. This gives

$$4 = 9 (e^2 - 1)$$

or
$$e^2 = \frac{13}{9}$$

or
$$e = \frac{\sqrt{13}}{3} .$$

11.3 EXERCISE

Short Answer Type

1. Find the equation of the circle which touches the both axes in first quadrant and whose radius is *a*.

2. Show that the point (x, y) given by $x = \dfrac{2at}{1+t^2}$ and $y = \dfrac{a(1-t^2)}{1+t^2}$ lies on a circle

   for all real values of *t* such that $-1 \leq t \leq 1$ where *a* is any given real numbers.

3. If a circle passes through the point $(0, 0)$ $(a, 0)$, $(0, b)$ then find the coordinates of its centre.

4. Find the equation of the circle which touches *x*-axis and whose centre is $(1, 2)$.

5. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then find the radius of the circle.

   [**Hint:** Distance between given parallel lines gives the diameter of the circle.]

6. Find the equation of a circle which touches both the axes and the line $3x - 4y + 8 = 0$ and lies in the third quadrant.

   [**Hint:** Let *a* be the radius of the circle, then $(-a, -a)$ will be centre and perpendicular distance from the centre to the given line gives the radius of the circle.]

7. If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is $(3, 4)$, then find the coordinate of the other end of the diameter.

8. Find the equation of the circle having $(1, -2)$ as its centre and passing through $3x + y = 14$, $2x + 5y = 18$

9. If the line $y = \sqrt{3}\ x + k$ touches the circle $x^2 + y^2 = 16$, then find the value of *k*.

   [**Hint:** Equate perpendicular distance from the centre of the circle to its radius].

10. Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double of its area.

    [**Hint:** concentric circles have the same centre.]

11. If the latus rectum of an ellipse is equal to half of minor axis, then find its eccentricity.

12. Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the eccentricity and foci.

13. If the eccentricity of an ellipse is $\dfrac{5}{8}$ and the distance between its foci is 10, then find latus rectum of the ellipse.

14. Find the equation of ellipse whose eccentricity is $\dfrac{2}{3}$, latus rectum is 5 and the centre is (0, 0).

15. Find the distance between the directrices of the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{20} = 1$.

16. Find the coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4.

17. Find the length of the line-segment joining the vertex of the parabola $y^2 = 4ax$ and a point on the parabola where the line-segment makes an angle θ to the x-axis.

18. If the points (0, 4) and (0, 2) are respectively the vertex and focus of a parabola, then find the equation of the parabola.

19. If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$ then find the value of m.
[**Hint:** Solving the equation of line and parabola, we obtain a quadratic equation and then apply the tangency condition giving the value of m].

20. If the distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then obtain the equation of the hyperbola.

21. Find the eccentricity of the hyperbola $9y^2 - 4x^2 = 36$.

22. Find the equation of the hyperbola with eccentricity $\dfrac{3}{2}$ and foci at ($\pm$ 2, 0).

Long Answer Type

23. If the lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 square units, then obtain the equation of the circle.

24. Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$.

25. Find the equation of a circle whose centre is (3, –1) and which cuts off a chord of length 6 units on the line $2x - 5y + 18 = 0$.

[**Hint:** To determine the radius of the circle, find the perpendicular distance from the centre to the given line.]

26. Find the equation of a circle of radius 5 which is touching another circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at (5, 5).

27. Find the equation of a circle passing through the point (7, 3) having radius 3 units and whose centre lies on the line $y = x - 1$.

28. Find the equation of each of the following parabolas
(a) Directrix $x = 0$, focus at (6, 0)   (b) Vertex at (0, 4), focus at (0, 2)
(c) Focus at (–1, –2), directrix $x - 2y + 3 = 0$

29. Find the equation of the set of all points the sum of whose distances from the points (3, 0) and (9, 0) is 12.

30. Find the equation of the set of all points whose distance from (0, 4) are $\dfrac{2}{3}$ of their distance from the line $y = 9$.

31. Show that the set of all points such that the difference of their distances from (4, 0) and (– 4, 0) is always equal to 2 represent a hyperbola.

32. Find the equation of the hyperbola with

   (a) Vertices (± 5, 0), foci (± 7, 0)   (b) Vertices (0, ± 7), $e = \dfrac{4}{3}$

   (c) Foci (0, ± $\sqrt{10}$), passing through (2, 3)

Objective Type Questions

State Whether the statements in each of the Exercises from 33 to 40 are True or False. Justify

33. The line $x + 3y = 0$ is a diameter of the circle $x^2 + y^2 + 6x + 2y = 0$.

34. The shortest distance from the point (2, –7) to the circle $x^2 + y^2 – 14x – 10y – 151 = 0$ is equal to 5.

   [**Hint:** The shortest distance is equal to the difference of the radius and the distance between the centre and the given point.]

35. If the line $lx + my = 1$ is a tangent to the circle $x^2 + y^2 = a^2$, then the point (l, m) lies on a circle.

   [**Hint:** Use that distance from the centre of the circle to the given line is equal to radius of the circle.]

36. The point (1, 2) lies inside the circle $x^2 + y^2 – 2x + 6y + 1 = 0$.

37. The line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$ if $ln = am^2$.

38. If P is a point on the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ whose foci are S and S′, then PS + PS′ = 8.

39. The line $2x + 3y = 12$ touches the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 2$ at the point (3, 2).

40. The locus of the point of intersection of lines $\sqrt{3}x – y – 4\sqrt{3}k = 0$ and

$\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different value of k is a hyperbola whose eccentricity is 2.

[**Hint:**Eliminate k between the given equations]

Fill in the Blank in Exercises from 41 to 46.

41. The equation of the circle having centre at $(3, -4)$ and touching the line $5x + 12y - 12 = 0$ is _____ .

[**Hint:** To determine radius find the perpendicular distance from the centre of the circle to the line.]

42. The equation of the circle circumscribing the triangle whose sides are the lines $y = x + 2$, $3y = 4x$, $2y = 3x$ is _____ .

43. An ellipse is described by using an endless string which is passed over two pins. If the axes are 6 cm and 4 cm, the length of the string and distance between the pins are _____.

44. The equation of the ellipse having foci $(0, 1)$, $(0, -1)$ and minor axis of length 1 is _____ .

45. The equation of the parabola having focus at $(-1, -2)$ and the directrix $x - 2y + 3 = 0$ is _____ .

46. The equation of the hyperbola with vertices at $(0, \pm 6)$ and eccentricity $\dfrac{5}{3}$ is _____ and its foci are _____ .

Choose the correct answer out of the given four options (M.C.Q.) in Exercises 47 to 59.

47. The area of the circle centred at $(1, 2)$ and passing through $(4, 6)$ is

(A) 5π (B) 10π (C) 25π (D) none of these

48. Equation of a circle which passes through $(3, 6)$ and touches the axes is

(A) $x^2 + y^2 + 6x + 6y + 3 = 0$ (B) $x^2 + y^2 - 6x - 6y - 9 = 0$

(C) $x^2 + y^2 - 6x - 6y + 9 = 0$ (D) none of these

49. Equation of the circle with centre on the y-axis and passing through the origin and the point $(2, 3)$ is

(A) $x^2 + y^2 + 13y = 0$ (B) $3x^2 + 3y^2 + 13x + 3 = 0$

(C) $6x^2 + 6y^2 - 13x = 0$ (D) $x^2 + y^2 + 13x + 3 = 0$

50. The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is

(A) $x^2 + y^2 = 9a^2$ (B) $x^2 + y^2 = 16a^2$

(C) $x^2 + y^2 = 4a^2$ (D) $x^2 + y^2 = a^2$

[**Hint:** Centroid of the triangle coincides with the centre of the circle and the radius of the circle is $\dfrac{2}{3}$ of the length of the median]

51. If the focus of a parabola is $(0, -3)$ and its directrix is $y = 3$, then its equation is

(A) $x^2 = -12y$ (B) $x^2 = 12y$ (C) $y^2 = -12x$ (D) $y^2 = 12x$

52. If the parabola $y^2 = 4ax$ passes through the point $(3, 2)$, then the length of its latus rectum is

(A) $\dfrac{2}{3}$ (B) $\dfrac{4}{3}$ (C) $\dfrac{1}{3}$ (D) 4

53. If the vertex of the parabola is the point $(-3, 0)$ and the directrix is the line $x + 5 = 0$, then its equation is

(A) $y^2 = 8\ (x + 3)$ (B) $x^2 = 8\ (y + 3)$

(C) $y^2 = -\ 8\ (x + 3)$ (D) $y^2 = 8\ (\ x + 5)$

54. The equation of the ellipse whose focus is $(1, -1)$, the directrix the line $x - y - 3 = 0$ and eccentricity $\dfrac{1}{2}$ is

(A) $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$

(B) $7x^2 + 2xy + 7y^2 + 7 = 0$

(C) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$

(D) none

55. The length of the latus rectum of the ellipse $3x^2 + y^2 = 12$ is

(A) 4   (B) 3   (C) 8   (D) $\dfrac{4}{\sqrt{3}}$

56. If e is the eccentricity of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a < b)$, then

(A) $b^2 = a^2\ (1 - e^2)$ (B) $a^2 = b^2\ (1 - e^2)$

(C) $a^2 = b^2\ (e^2 - 1)$ (D) $b^2 = a^2\ (e^2 - 1)$

57. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half of the distance between the foci is

(A) $\dfrac{4}{3}$ (B) $\dfrac{4}{\sqrt{3}}$ (C) $\dfrac{2}{\sqrt{3}}$ (D) none of these

58. The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$. Its equation is

(A) $x^2 - y^2 = 32$ (B) $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ (C) $2x - 3y^2 = 7$ (D) none of these

59. Equation of the hyperbola with eccentricty $\dfrac{3}{2}$ and foci at $(\pm 2, 0)$ is

(A) $\dfrac{x^2}{4} - \dfrac{y^2}{5} = \dfrac{4}{9}$ (B) $\dfrac{x^2}{9} - \dfrac{y^2}{9} = \dfrac{4}{9}$ (C) $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ (D) none of these